$$\sigma_n(\rho) = -\frac{P}{\pi^2}\frac{\sqrt{a^2-b^2}}{\sqrt{\rho^2-a^2}\,(\rho^2-b^2)}\left(\frac{b}{\rho}\right)^n, \quad \tau_n = 0$$

In the case of an isotropic body this latter holds for a Poisson's ratio of one-half.

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SOME PROBLEMS OF THE NONHOMOGENEOUS ELASTICITY THEORY

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Within the framework of the plane problem of the theory of elasticity, we consider the equilibrium of an elastic plane with thin different elastic inclusions, situated along a straight line. We give the formulation of the boundary value problems on the basis of the approach adopted from the theory of a thin airfoil. We present an effective method for obtaining the exact solution of a general class of problems of the above indicated type. We analyze the effect of the inclusions on the strength and we formulate criteria for the initiation of brittle fracture.

1. Formulation of the boundary value problem. In many materials which represent a practical interest, we frequently encounter thin elastic inclusions of a different material. Such are, for example, layers of graphite in cast iron, areas of oxidized metal in alloys, layers of low strength clay or sand in tectonic faults, welds, etc. The inclusions in the basic material lead to stress concentrations which affect essentially the strength properties of the material as a whole.

We consider the deformation of an unbounded, elastic, homogeneous, isotropic space with an arbitrary number of thin cylindrical inclusions of a different elastic material. Let the plane xy be some cross section of these cylinders. We assume that each of these inclusions has in the plane xy an axis of symmetry which coincides with the x -

axis (Fig. 1). In the problems to be examined we consider that the stresses and strains are independent of the coordinate z (a combination of plane strain and longitudinal shear, anti-plane strain).

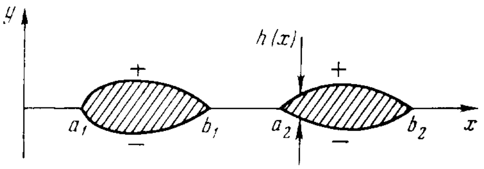

Fig. 1

We denote by $h(x)$ the thickness of the inclusion and we assume that the conditions of rigid contact hold at the contiguous points of the boundary of the basic material and of the inclusion. Usually, the operation of coupling of different materials is accompanied by the formation of stresses even in the absence of external loads; these will be called initial residual or technological stresses.

We assume that for each inclusion the following conditions hold:

$$h(x) \ll 2l, \quad |\partial h / \partial x| \ll 1 \tag{1.1}$$

where $2l$ is the length of the inclusion. In this case, for the effective solving of the elasticity theory problems, we can apply the following method, taken from the theory of a thin airfoil [1]: we discard the boundary conditions corresponding to the boundaries of the inclusions on the x-axis omitting the small quantities in the boundary conditions and we solve the obtained boundary value problem for the set L of the corresponding mathematical cuts along the x-axis. Adding to the initial coordinates of the material point its displacements, obtained from the approximate solution, we get the position of this point in the deformed state.

We will assume that the external load is applied only at infinity. Therefore, the equilibrium conditions of the thin inclusions have the form

$$[\sigma_y] = [\tau_{xy}] = [\tau_{xz}] = 0 \quad \text{on } L \tag{1.2}$$

Here $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are stresses; the quantity $[A]$ denotes the difference $A^+ - A^-$. The plus and minus signs correspond to the values of the corresponding functions at the upper and lower sides of the cuts L, i.e. at the upper and lower boundaries of the inclusions. Consequently, the stresses σ_y, τ_{xy} and τ_{xz} are continuous at the cuts L; therefore, they can be written with the indices plus and minus omitted.

Hooke's law for thin elastic inclusions can be written as

for $y = 0$, x on L

$$\sigma_y = \lambda_1 [v] + 2\lambda_1 v_0(x) \quad \tau_{xy} = \lambda_2 [u] + 2\lambda_2 u_0(x), \quad \tau_{yz} = \lambda_3 [w] + 2\lambda_3 w_0(x) \tag{1.3}$$

$$\left(\lambda_1 = \frac{E_1}{h(x)}, \quad \lambda_2 = \frac{\mu_2}{h(x)}, \quad \lambda_3 = \frac{\mu_3}{h(x)}\right)$$

Here E_1, μ_2, μ_3 are the Young's modulus and the shear moduli of the orthotropic inclusions, characterizing its strain normal to the cut, the transverse and longitudinal shear strains, respectively; u, v, w are the components of the displacement vector of the material point in the basic material; $2u_0$, $2v_0$, $2w_0$ are the components of the given discontinuity of the displacement along the cuts L (residual effect).

The Eqs. (1.3) represent the boundary conditions of the formulated problem. If the exterior load is equal to zero, then we obtain the problem on the determination of the initial stresses occurring because of the technological (residual) stress.

For example, we assume that at the beginning thin cavities were cut out from the elastic body and then inclusions from a different elastic material, coinciding exactly in form with the corresponding cavities but having a lower temperature than the basic material, have been inserted in these cavities. Obviously, after the leveling of the temperatures, the residual effect is

$$2v_0 = 2\alpha h \Delta T, \quad 2u_0 = 2w_0 = 0$$

Here ΔT is the difference of the initial temperatures and α is the temperature coefficient of linear expansion.

The quantities λ_1, λ_2 and λ_3 are similar to the elastic spring constants in Winkler's foundation theory.

The solution of the boundary value problem (1.3) can be represented [2] in the form of the sum of the solutions of the following three boundary value problems for the upper half-plane corresponding to normal tension, transverse shear and longitudinal shear.

1°. Normal tension

$$\begin{aligned} &\text{for} \quad y = 0, \quad x \text{ on } L \quad \sigma_y = 2\lambda_1 v^+ + 2\lambda_1 v_0 \\ &\text{for} \quad y = 0 \quad \tau_{xy} = \tau_{yz} = 0 \end{aligned} \tag{1.4}$$

2°. Transverse shear

$$\begin{aligned} &\text{for} \quad y = 0, \quad x \text{ on } L \quad \tau_{xy} = 2\lambda_2 u^+ + 2\lambda_2 u_0 \\ &\text{for} \quad y = 0 \quad \sigma_y = \tau_{yz} = 0 \end{aligned} \tag{1.5}$$

3°. Longitudinal shear

$$\begin{aligned} &\text{for} \quad y = 0, \quad x \text{ on } L \quad \tau_{xz} = 2\lambda_3 w^+ + 2\lambda_3 w_0 \\ &\text{for} \quad y = 0 \quad \sigma_y = \tau_{xy} = 0 \end{aligned} \tag{1.6}$$

The given stresses at infinity can also be always represented in the form of the superposition of stresses, symmetric and antisymmetric with respect to the x-axis. The boundary conditions (1.4) and (1.5) correspond to plane strain, where in the Case 1° the displacement v, while in Case 2° the displacement u, are odd functions of y. In Case 3° (longitudinal shear) the displacement is an odd function of y.

We give the Kolosov-Muskhelishvili [3] representation in terms of the complex potentials $\Phi(z)$, $\Psi(z)$ and $f(z)$, which are analytic functions of the complex variable $z = x + iy$ in the domain occupied by the body, i.e. in the exterior of the cuts L

$$\begin{aligned} &\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z) \\ &\sigma_y - i\tau_{xy} = \Phi(z) + \overline{\Phi(z)} + \overline{\Omega(z)} + (z - \bar{z})\ \Phi'(z) \\ &2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa\Phi(z) - \overline{\Phi(z)} - \overline{\Omega(z)} - (z - \bar{z})\Phi'(z) \\ &w = \operatorname{Re} f(z), \quad \tau_{xz} + i\tau_{yz} = \mu\overline{f'(z)} \\ &\Omega(z) = z\Phi'(z) + \Psi(z) \end{aligned} \tag{1.7}$$

Here μ and ν are the shear modulus and Poisson's ratio of the basic material $\varkappa = 3 - 4\nu$

for the state of plane strain and $\varkappa = (3 - \nu) / (1 + \nu)$ for the state of plane stress.

For the sake of simplicity, we will assume that the stresses at infinity are equal to zero. The method of superposition allows us to reduce to this case the problem with an arbitrary nonzero state of stress at infinity. In this case, we can obtain, with the aid of the representation (1.7), the following formulas for the problems (1.4)–(1.6):

1°. Normal tension

$$f'(z) = 0, \quad \Omega(z) = 0$$
$$\text{for} \quad y = 0 \qquad \sigma_x = \sigma_y = 2\,\mathrm{Re}\,\Phi(z), v = \frac{\varkappa + 1}{2\mu}\,\mathrm{Im}\,\varphi(z) \tag{1.8}$$

2°. Transverse shear

$$f'(z) = 0, \quad 2\Phi(z) + \Omega(z) = 0$$
$$\text{for} \quad y = 0 \qquad \tau_{xy} = \mathrm{Im}\ \Omega(z), u = -\frac{\varkappa + 1}{4\mu}\,\mathrm{Re}\,\omega(z) \tag{1.9}$$

3°. Longitudinal shear

$$\Phi(z) = 0, \qquad \Omega(z) = 0$$
$$\text{for} \quad y = 0 \qquad \tau_{yz} = -\mu\,\mathrm{Im}\,f'(z), \quad w = \mathrm{Re}\,f(z) \tag{1.10}$$

Here

$$\omega'(z) = \Omega(z), \qquad \varphi'(z) = \Phi(z) \tag{1.11}$$

Obviously, any linear boundary value problem of the theory of elasticity can be reduced to the linear combination of normal tension, longitudinal shear and transverse shear, if the boundary of the body is situated along the x-axis. The indicated method for the decomposition of any boundary value problem of this type into the sum of three problems (for the normal tension, the longitudinal and transverse shear) is expecially convenient for the solving of concrete problems, since the mathematical problems for each of these problems are equivalent. It is sufficient to obtain, for example, the solution for the normal tension; the solutions for the other cases are obtained with the aid of obvious substitutions. Therefore, in the sequel we will restrict ourselves only to the examination of the normal tension.

Making use of the relations (1.8), from the boundary conditions (1.4) we obtain the following boundary value problem for the determination of the function $\varphi(z)$:

$$\text{for} \quad y = 0, \quad x \ \text{on} \ L$$
$$\mathrm{Re}\,\varphi'(z) = \frac{\varkappa + 1}{2\mu}\,\lambda_1(x)\,\mathrm{Im}\,\varphi(z) + \lambda_1(x)\,v_0(x) \tag{1.12}$$
$$\text{for} \quad y = 0, \quad x \ \text{outside} \ L \quad \mathrm{Im}\varphi(z) = 0$$

The solution in closed form of the boundary value problem (1.12) for and arbitrary function $\lambda_1(x)$ is unattainable. Apparently, Poincaré was the first who has encountered problems of this type in the solving of some hydrodynamic problems of the theory of tides.

We restrict ourselves to the solution of some classes of boundary value problems (1.12); these solutions can be found in closed form and they include practically the most important cases.

2. The effective solution of a general class of boundary value problems. We consider the boundary value problem (1.12) with the coefficient $\lambda_1(x)$ of the following form:

$$\lambda_1(x) = iX^+(x)\frac{Q(x)}{P(x)} \tag{2.1}$$

Here $P(x)$ and $Q(x)$ are arbitrary polynomials with real coefficients and the function $X^+(x)$ is the value of the function $X(z)$, analytic outside L, on the upper sides of the cuts

$$X(z) = \prod_{i=1}^{n} \sqrt{(z-a_i)(z-b_i)} \qquad (X(z) \to z^n \quad \text{for} \quad z \to \infty)$$

(n is the number of cuts, a_i and b_i are the abscissas of the left-hand and right-hand ends of the i-th cut). The function $X^+(x)$ is imaginary on L and real outside L. Therefore the new function

$$F(z) = \varphi'(z) - \frac{(\varkappa+1)}{2\mu} X(z)\frac{Q(z)}{P(z)}\varphi(z) \tag{2.2}$$

is analytic outside L, except at the zeros of the polynomial $P(z)$ (where it has poles of appropriate order) and possibly at a point at infinity.

For the functions $\lambda_1(x)$ of the form (2.1), the boundary value problem (1.12) takes the form

$$\text{for} \quad y=0, \quad x \text{ on } L \quad \operatorname{Re} F(z) = \lambda_1(x)v_0(x) \tag{2.3}$$

$$\text{for} \quad y=0, \quad x \text{ outside } L \quad \operatorname{Im} F(z) = 0$$

The solution of this problem is given by the Keldysh-Sedov formula, modified somewhat in the case of the presence of poles for the unknown function $F(z)$. Then, the function $\varphi(z)$ is determined from the ordinary linear differential equation (2.2) of the first order. To avoid the overloading of the presentation, we carry out the more detailed computations only for the case of a single inclusion.

Let us assume that the thickness of the elastic inclusion varies according to the law

$$h(x) = \frac{2P(x)}{i\sqrt{x^2-l^2}\,Q(x)} \qquad (-l<x<l) \tag{2.4}$$

Here $P(x)$ and $Q(x)$ are some polynomials with real coefficients.

The analytic function

$$F(z) = \varphi'(z) - \frac{(\varkappa+1)E_1Q(z)}{2\mu P(z)}\sqrt{z^2-l^2} \tag{2.5}$$

must satisfy the following boundary conditions:

$$\text{for} \quad y=0, \quad |x|<l \quad \operatorname{Re} F(z) = \lambda_1(x)\,v_0(x) \tag{2.6}$$

$$\text{for} \quad y=0, \quad |x|>l \quad \operatorname{Im} F(z) = 0$$

The solution of the boundary value problem (2.6) can be written in the following form:

$$F(z) = \frac{1}{\pi i\sqrt{z^2-l^2}}\int_{-l}^{+l}\frac{\lambda_1(x)\,v_0(x)\sqrt{x^2-l^2}}{x-z}\,dx + \frac{L(z)}{P(z)\sqrt{z^2-l^2}}$$

where $L(z)$ is some polynomial with real coefficients. Solving the first order differential equation (2.5) with respect to $\varphi(z)$, we find

$$\varphi(z) = \frac{1}{\varphi_0(z)}\int F(z)\,\varphi_0(z)\,dz_0 \tag{2.7}$$

Here

$$\varphi_0(z) = \exp\left\{-\int \frac{(\varkappa+1)E_1\sqrt{z^2-l^2}}{2\mu P(z)} Q(z)\,dZ\right\}$$

The polynomial $L(z)$ is determined from the condition of the vanishing of the function $\varphi(z)$ at infinity (since the resultant of the forces applied to the inclusion is assumed to be zero) and form the condition of the analyticity ofthe function $\varphi(z)$ at the zeros of the polynomial $P(z)$.

With a function of type (2.4) we can approximate, with any degree of accuracy, any continuous function $h(x)$ on any finite interval, if it vanishes at the extremities of the interval. Therefore, the solution of the class of problems under consideration can be used as an approximate effective method of solution also in the general case. (This method has been indicated in the book [2]).

By a simple transformation, the case of a nonhomogeneous inclusion, when $E_1 = E_1(x)$, can be reduced to the preceding one.

The case of a periodic system of inclusions along the x-axis also presents interest. In this case, an effective solution of the boundary value problem (1.12) can be obtained for coefficients $\lambda_1(x)$ of the form:

$$\lambda_1(x) = iX^+(x)\frac{Q(\sin x)}{P(\sin x)}$$

The functions $X(z)$ and $F(z)$ are as follows:

$$X(z) = \sqrt{\sin^2 \pi z/2L - \sin^2 \pi l/2L}$$

$$F(z) = \varphi'(z) - \frac{\varkappa+1}{2\mu} X(z)\frac{Q(\sin z)}{P(\sin z)}\varphi(z)$$

Here $2l$ is the length of one inclusion and $2L$ is the period.

The boundary value problem (2.3) for a periodic system of cuts L, with the aid of the conformal mapping

$$w = \sin \pi z/2L \tag{2.8}$$

reduces to the already considered case of one cut in the plane w

3. Semi-infinite inclusion. 1°. An elastic inclusion in the form of a thin wedge. Let us assume that the thickness of the inclusion varies according to the law

$$h(x) = -2\alpha x \qquad (\alpha \ll 1,\ x < 0) \tag{3.1}$$

where α is the opening angle of the wedge. In this case the function

$$\lambda_1(x) = E_1 / 2\alpha x \tag{3.2}$$

and the corresponding boundary value problem for the determination of the function $\varphi(z)$ has the form

$$\begin{aligned} &\text{for } y=0,\ x<0 \qquad \operatorname{Re}\varphi'(z) = -\frac{(\varkappa+1)E_1}{4\mu\alpha x}\operatorname{Im}\varphi(z) \\ &\text{for } y=0,\ x>0 \qquad \operatorname{Im}\ \varphi(z) = 0 \end{aligned} \tag{3.3}$$

The boundary value problem (3.3) admits the following group of transformations:

$$x' = C_1 x, \qquad \varphi' = C_2\varphi \tag{3.4}$$

where C_1 and C_2 are arbitrary real parameters. Therefore, in the case under consideration, the solution of the boundary value problem (3.3) has the form [2]

$$\varphi(z) = Az^{\lambda} \tag{3.5}$$

where A is an arbitrary real coefficient. Substituting (3.5) into (3.3), we obtain the following characteristic equation for the determination of λ:

$$\frac{4\mu\alpha\lambda}{(\varkappa+1)E_1} = -\operatorname{tg}\pi\lambda \tag{3.6}$$

The roots of the Eq. (3.6), as is obvious from the graphic representation of the solution, are situated on the segments $(1/2,\ 1)$, $(3/2,\ 2)$, . . .$(-1/2,\ -1)$, $(-3/2,\ -2)$ . . . of the real axis, one in each of these segments. As it follows from the theorem on homogeneous solutions [2], the solution of the correct boundary value problem corresponds to a unique root, situated on the segment $(1/2,\ 1)$. The dependence of the magnitude of this root on the dimensionless elasticity coefficient of the inclusion

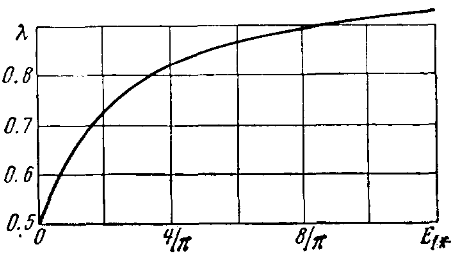

Fig. 2

$$E_{1*} = \frac{(\varkappa+1)E_1}{4\mu\alpha}$$

is represented in Fig. 2. As generally for problems of class N, the coefficient A is assumed to be given in advance [2].

Analogous self-similar solution takes place for an elastic inclusion in the form of a wedge with an arbitrary opening angle 2α, and also for an arbitrary number of different inclusions of this type. In each of these problems we obtain its own transcendental equation for the determination of the number λ.

2°. An elastic inclusion of parabolic form. Assume that the thickness of the inclusion varies according to the law

$$h(x) = \beta\ |\sqrt{x}\,| \qquad (x < 0)$$

where β is a specified real thickness parameter.

The function $F(z)$

$$F(z) = \varphi'(z) - \frac{(\varkappa+1)E_1}{2\mu\sqrt{z}\,\beta}\varphi(z) \tag{3.7}$$

must satisfy the following boundary conditions:

$$\text{for } y = 0,\ x < 0 \quad \operatorname{Re} F(z) = 0$$
$$\text{for } y = 0,\ x > 0 \quad \operatorname{Im} F(z) = 0$$

The solution of this boundary value problem is determined except for the arbitrary real factors K_{I} and B

$$F(z) = \frac{K_{\mathrm{I}}}{2\sqrt{2\pi z}} + B\sqrt{z}$$

Solving Eq. (3.7) with respect to $\varphi(z)$ and determining B, we find (it is assumed that for $x \to \infty$, $y = 0$ we have $\sigma_x = \sigma_y$)

$$\varphi(z) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi}}\left[\sqrt{z} + \frac{(\varkappa+1)E_1}{2\mu\beta}z\right]$$

4. Finite inclusions of a particular form. We assume that along the contour of the elastic inclusion there exists some specified jump in the displacement, for example because of the difference of the initial temperatures of the inclusion and of the basic material; at infinity the stress σ_y is constant and equal to σ_y^∞, while the remaining stresses are equal to zero. In the case under consideration the boundary value problem has the form

$$\text{for} \quad y=0, \ |x|<l \tag{4.1}$$

$$\operatorname{Re}\varphi'(z) = \frac{\varkappa+1}{2\mu}\lambda_1(x)\operatorname{Im}\varphi(z) + E_1\alpha\Delta T - \frac{1}{4}\sigma_y^\infty$$

$$\text{for} \quad y=0, \quad |x|>l \ \operatorname{Im}\varphi(z)=0$$

$$\text{for} \quad z\to\infty \quad \varphi'(z) = {}^1/_4\sigma_y^\infty$$

For $y=0$ the following formulas hold:

$$v = \frac{\varkappa+1}{2\mu}\operatorname{Im}\varphi(z)$$

$$\sigma_x = 2\operatorname{Re}\varphi'(z) - {}^1/_2\ \sigma_y^\infty, \qquad \tau_{xy}=0$$

$$\sigma_y = 2\operatorname{Re}\varphi'(z) + {}^1/_2\sigma_y^\infty, \qquad \tau_{xz}=\tau_{yz}=0$$

1°. A single elliptic inclusion. We assume that the thickness of the inclusion varies according to the law (β is a specified real thickness parameter)

$$h(x) = \beta\left|\sqrt{x^2-l^2}\right|, \quad \beta \ll 1 \quad (-l<x<l) \tag{4.2}$$

The auxiliary function

$$F(z) = \varphi'(z) - \frac{(\varkappa+1)E_1\varphi(z)}{2\mu\beta\sqrt{z^2-l^2}} \tag{4.3}$$

must satisfy the following boundary conditions:

$$\text{for} \quad y=0, \quad |x|<l \tag{4.4}$$

$$\operatorname{Re} F(z) = E_1\alpha\Delta T - {}^1/_4\sigma_y^\infty$$

$$\text{for} \quad y=0, \quad |x|>l \quad \operatorname{Im} F(z) = 0$$

$$F(z) = \frac{1}{4}\sigma_y^\infty\left[1-\frac{(\varkappa+1)E_1}{2\mu\beta}\right] \quad \text{for} \quad z\to\infty$$

The solution of the boundary value problem (4.4) has the following form:

$$F(z) = a - \frac{bz}{\sqrt{z^2-l^2}} \tag{4.5}$$

$$a = \alpha E_1\Delta T - \frac{1}{4}\sigma_y^\infty, \qquad b = \alpha E_1\Delta T - \frac{1}{2}\sigma_y^\infty + \frac{(\varkappa+1)E_1\sigma_y^\infty}{8\mu\beta}$$

Solving Eq. (4.3), we obtain the function $\varphi(z)$ in the form (2.7), where the integral is taken between the limits l and z, the function $F(z)$ is given by the formula (4.5), while $\varphi_0(z)$ is as follows:

$$\varphi_0(z) = \left(z+\sqrt{z^2-l^2}\right)^{-\lambda}, \quad \lambda = \frac{(\varkappa+1)E_1}{2\mu\beta} \tag{4.6}$$

With the aid of the formulas (4.1) and (4.6) we find the stress σ_y

$$\text{for} \quad y=0, \ x>l$$

$$\sigma_y = 2\alpha E_1\Delta T - \frac{2bx}{\sqrt{x^2-l^2}} + \frac{2\lambda}{\sqrt{x^2-l^2}\,\varphi_0(x)}\int_l^x F(x)\varphi_0(x)\,dx$$

We note the following formulas:

$$\text{for} \quad z \to l \quad \varphi'(z) = -\frac{b\sqrt{l}}{\sqrt{2(z-l)}} + O(1) \tag{4.7}$$

$$\text{for} \quad y = 0, \quad x \to l + \varepsilon_1 \quad \sigma_y = -\frac{b\sqrt{2l}}{\sqrt{x-l}} \quad (\varepsilon_1 \ll l)$$

Comparing (4.7) and (3.12), we obtain

$$K_I = -2b\sqrt{\pi l}$$

2°. A periodic system of inclusions. We consider an elastic body with a periodic system of inclusions of the following form (β is a constant, $\beta \ll 1$):

$$h(x) = \beta l \left| \sqrt{\sin^2 \pi x / 2L - \sin^2 \pi l / 2L} \right| \tag{4.8}$$

We assume that along the cuts there exists some initial constant jump in the displacements, just as in the previous problem; at infinity the constant stress σ_y is equal to σ_y^∞, while the remaining stresses are equal to zero. With the aid of the transformation (2.8) we switch from the physical plane z to the parametric plane of the complex variable w. The exterior of the periodic system of cuts in the plane z corresponds in a one-to-one manner to the infinite sheeted Riemann surface w with a cut along

$$(-\sin \pi l / 2L, \ \sin \pi l / 2L)$$

Making use of the method of solution of Sect. 2 and of the results obtained for the inclusions, the solution of the problem under consideration can be written in the form (2.7), where the integral is taken between the limits l and z and

$$F(z) = a - b \sin \frac{\pi z}{2L} \left(\sin^2 \frac{\pi z}{2L} - \sin^2 \frac{\pi l}{2L} \right)^{-1/2} \tag{4.9}$$

$$\varphi_0(z) = \exp\left\{ -\frac{(\varkappa+1)E_1}{2\mu\beta l} \int_l^z \left(\sin^2 \frac{\pi z}{2L} - \sin \frac{\pi l}{2L} \right)^{-1/2} \right\}$$

$$a = \alpha E_1 \Delta T - \frac{1}{4}\sigma_y^\infty, \quad b = \alpha E_1 \Delta T - \frac{1}{2}\sigma_y^\infty + \frac{(\varkappa+1)E_1\sigma_y^\infty}{8\mu\beta l}$$

$$\text{for} \quad z \to l \quad \varphi'(z) = -b\sqrt{\frac{L}{\pi} \operatorname{tg} \frac{\pi l}{2L}} \frac{1}{\sqrt{z-l}}$$

$$K_I = -2b\sqrt{2L \operatorname{tg} \pi l / 2L}$$

3°. Sharp inclusions. Assume that the thickness of a single inclusion varies according to the law (an elongated oval with pointed ends)

$$h(x) = \beta l^{-2} \left| (l^2 - x^2)^{3/2} \right| \qquad (\beta \ll 1) \tag{4.10}$$

As before, we consider as given: on the cut $(-l, +l)$ the initial (residual) strain due to the difference of the inital temperatures, while at infinity, the constant stress σ_y^∞. The solution of this problem, obtained by the general method of Sect. 2, has the form (2.7), where the integral is taken between the limits l and z, while

$$F(z) = b + \frac{1}{4}\sigma_y^\infty - \frac{bz}{\sqrt{z^2 - l^2}} \qquad \left(b = \alpha E_1 \Delta T - \frac{1}{2}\sigma_y^\infty \right)$$

$$\varphi_0(z) = \exp\left\{-\frac{(\varkappa+1)E_1 z}{2\mu\beta\sqrt{z^2-l^2}}\right\}$$

$$\text{for } z \to l \quad \varphi'(z) = \frac{4\sqrt{2}\mu^2\beta^2 b}{(\varkappa+1)^2 E_1^2\sqrt{l}} + O(\sqrt{z})$$

Obviously, the presence of the cusp at the inclusion removes the singularity at the extremity of the inclusion.

5. An elastic inclusion of constant thickness. Let us assume that a homogeneous elastic inclusion has a constant thickness

$$h(x) = h = \text{const} \tag{5.1}$$

We will assume that along the contour of the elastic inclusion there exists some given constant initial displacement $2v_0$. At infinity the stress σ_y is constant and equal to σ_y^∞, while the remaining stresses are equal to zero.

The fundamental relations and the boundary value problem in this case take the form:

$$\text{for } y=0 \quad v = \frac{\varkappa+1}{2\mu}\operatorname{Im}\varphi(z), \quad \sigma_y - \sigma_x = \sigma_y^\infty \tag{5.2}$$

$$\sigma_y = 2\operatorname{Re}\varphi'(z) = {}^1/_2\sigma_y^\infty$$

$$\text{for } y=0, \quad |x| < l \tag{5.3}$$

$$\operatorname{Re}\varphi'(z) = \frac{\varepsilon}{h}\operatorname{Im}\varphi(z) + \delta$$

$$\text{for } y=0, \quad |x| > l \quad \operatorname{Im}\varphi(z) = 0$$

$$\text{for } z \to \infty \quad \varphi'(z) = {}^1/_4\sigma_y^\infty$$

Here

$$\varepsilon = \frac{E_1(\varkappa+1)}{2\mu}, \qquad \delta = \frac{E_1}{h}v_0 - \frac{1}{4}\sigma_y^\infty$$

The constant ε is not equal to zero at the endpoints of the interval $(-l, +l)$, therefore it cannot be approximated by a linear combination of functions of the type (2.4). For the solving of the boundary value problem (5.3) we apply the asymptotic method of expansion with respect to small and large values of the parameter ε (the method is explained in [2]). This method is effective also in more general cases of nonhomogeneous problems of the theory of elasticity and not only for thin inclusions.

1°. The solution for small ε. In the case $\varepsilon \leqslant 1$, we seek the solution of the boundary value problem (5.3) in the form

$$\varphi(z) = \varphi_0(z) + \varepsilon\varphi_1(z) + \varepsilon^2\varphi_2(z) + \ldots \tag{5.4}$$

Here $\varphi_0, \varphi_1, \varphi_2, \ldots$ are unknown functions. Substituting (5.4) into the condition (5.3), we can obtain the following chain of standard Dirichlet boundary value problems on the cut $(-l, +l)$:

$$\text{(0)} \quad \text{for } y=0, \quad |x|<l \quad \operatorname{Re}\varphi_0'(z) = \delta \tag{5.5}$$

$$\text{for } z\to\infty \quad \varphi_0'(z) = {}^1/_4\sigma_y^\infty$$

$$\text{(1)} \quad \text{for } y=0, \quad |x|<l \quad \operatorname{Re}\varphi_1'(z) = \frac{1}{h}\operatorname{Im}\varphi_0(z)$$

$$\text{for } z\to\infty \quad \varphi_1(z) \to 0$$

$$\text{(2)} \quad \text{for } y=0, \quad |x|<l \quad \operatorname{Re}\varphi_2'(z) = \frac{1}{h}\operatorname{Im}\varphi_1(z) \quad \text{for } z\to\infty \quad \varphi_2(z)\to 0$$

and so on. The solution of these problems can be easily found successively, applying the known Keldysh-Sedov formula.

2°. The solution for large ε. In the case $\varepsilon \gg 1$, we seek the solution of the boundary value problem (5.3) in the form

$$\varphi(z) = \varphi_0(z) + \frac{1}{\varepsilon}\varphi_1(z) + \frac{1}{\varepsilon^2}\varphi_2(z) \tag{5.6}$$

Substituting (5.6) into the condition (5.3), we obtain easily the following chain of standard Dirichlet boundary value problems for the half-plane:

$$\begin{aligned}
&(0)\quad \text{for}\quad y = 0 \quad \operatorname{Im}\varphi_0 = 0\\
&\qquad \text{for}\quad z \to \infty \quad \varphi_0'(z) = {}^1/_4\sigma_y^\infty\\
&(1)\quad \text{for}\quad y = 0, \quad |x| < l\\
&\qquad \operatorname{Im}\varphi_1(z) = h\operatorname{Re}\varphi_0' - \delta h\\
&\qquad \text{for}\quad y = 0, \quad |x| > l \quad \operatorname{Im}\varphi_1(z) = 0\\
&\qquad \text{for}\quad z \to \infty \quad \varphi_1(z) \to 0\\
&(2)\quad \text{for}\quad y = 0, \quad |x| < l \quad \operatorname{Im}\varphi_2(z) = h\operatorname{Re}\varphi_1'(z)\\
&\qquad \text{for}\quad y = 0, \quad |x| > l \quad \operatorname{Im}\varphi_2(z) = 0\\
&\qquad \text{for}\quad z \to \infty \quad \varphi_2(z) \to 0
\end{aligned} \tag{5.7}$$

and so on. The solution of these problems can be easily found successively, one after another.

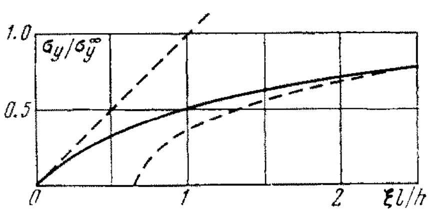

Fig. 3

In order to find the solution for intermediate values of ε we apply a combination of different asymptotic expansions for small and large ε. In some cases, practically good (i.e. very close to exact) results are obtained if we restrict ourselves to the first two-three terms of the expansion with respect to ε and $1/\varepsilon$. We note that similar methods are widely used in the theory of boundary layers.

Restricting ourselves to the first approximation, the solution can be written as:

$$\text{for}\quad \varepsilon \ll 1 \quad \varphi'(z) = \delta - \frac{\delta - {}^1/_4\sigma_y^\infty}{\sqrt{z^2 - l^2}}\left(-z - \frac{2l}{\pi h}\varepsilon z + \varepsilon\frac{z^2 - l^2}{\pi h}\ln\frac{z - l}{z + l}\right) \tag{5.8}$$

$$\text{for}\quad z \to l, \quad \varepsilon \ll 1 \quad \varphi'(z) = \frac{l\left(\delta - \frac{1}{4}\sigma_y^\infty\right)\left(-1 + \frac{2\varepsilon l}{\pi h}\right)}{\sqrt{2l(z - l)}}$$

$$\text{for}\quad \varepsilon \gg 1 \quad \varphi'(z) = \frac{1}{4}\sigma_y^\infty + \frac{2lh\,({}^1/_4\sigma_y^\infty - \delta)}{\pi\varepsilon(z^2 - l^2)} \tag{5.9}$$

In particular, the stress σ_y at the point $y = 0$, $x = 0$ of the inclusion is

$$\sigma_y = \begin{cases} 2\frac{E_1}{h}v_0 - \frac{2l\varepsilon}{h}\left(\frac{E_1 v_0}{h} - \frac{1}{2}\sigma_y^\infty\right) & \text{for}\quad \varepsilon \ll 1\\ \sigma_y^\infty - \frac{2h}{\pi l\varepsilon}\left(\sigma_y^\infty - 2\frac{E_1}{h}v_0\right) & \text{for}\quad \varepsilon \gg 1 \end{cases} \tag{5.10}$$

On Fig. 3 the continuous line represents the approximate dependence of the dimensionless stress $\sigma_y / \sigma_y^\infty$ at the point $x = 0, \ y = 0$ of the inclusion in the case $v_0 = 0$ for different intermediate values of the dimensionless parameter $\varepsilon l / h$, obtained by the combination of the asymptotics. The broken lines correspond to the asymptotic formulas (5.10).

6. The analysis of the brittle fracture of bodies with thin elastic inclusions. According to the general approach [2], the criteria for the initiation of brittle fracture in a body with thin inclusions are formulated in the following way: the principal coefficient in the expansion of the stress function at the endpoint of the inclusion must attain at the moment of local fracture some constant of the given constituent material. This constant may depend only on the strength of the basic material and of the inclusions and also on the strength of the coupling and on the form of the inclusions at its end; however, it does not depend on the form of the body, on the exterior loads and on other similar factors.

Making use of the solutions of the particular problems, obtained in Sects. 3–5, and of the criteria of local fracture, the conditions for the absence of fracture can be written in the form of the following inequalities:

one elliptic inclusion (formulas (4.6) and (4.7))

$$\sigma_y^\infty - 2\alpha E_1 \Delta T - \frac{(\varkappa + 1) E_1 \sigma_y^\infty}{4\mu\beta} < \frac{K_{1c}}{\sqrt{\pi l}}$$

a periodic system of inclusions (formula (4.9))

$$\sigma_y^\infty - 2\alpha E_1 \Delta T - \frac{(\varkappa + 1) E_1 \sigma_y^\infty}{4\mu\beta l} < \frac{K_{1c}}{\sqrt{2L \operatorname{tg} \pi l / 2L}}$$

one pointed inclusion (formula (4.10))

$$-\frac{8\sqrt{2}\,\mu^2\beta^2 b}{(\varkappa + 1)^2 E_1^2 \sqrt{l}} < \sigma_c$$

one inclusion of constant thickness (formulas (5.8) and (5.9))

$$\text{for } \varepsilon \ll 1 \quad \left(\sigma_y^\infty - 2\frac{E_1}{h} v_0\right)\left(1 - \frac{2\varepsilon l}{\pi h}\right) < \frac{K_{2c}}{\sqrt{\pi l}}$$

$$\text{for } \varepsilon \gg 1 \quad \frac{h}{\pi\varepsilon}\left(\frac{1}{2}\sigma_y^\infty - \frac{E_1}{h} v_0\right) < K_{3c} \qquad (\varepsilon = E_1(\varkappa + 1) / (2\mu))$$

Here K_{1c}, K_{2c}, K_{3c}, σ_c are constants of the constituent material which have to be determined experimentally.

We note that the constant K_{3c} has the dimension of a force multiplied by length, while the constants K_{1c} and K_{2c} have the dimension of a force multiplied by length to the power $^3/_2$. It should be stressed that for inclusions with smoothly rounded or trimmed ends the obtained singularities correspond in fact to some intermediate asymptotics of the exact solution, valid at distances from the endpoint of the inclusion which are large in comparison with the radius of the rounding or with the thickness of the inclusion, but small in comparison with the length of the inclusion.

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CRACK PROPAGATION AT VARIABLE VELOCITY

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The plane problem of rectilinear crack propagation in an elastic medium subjected to arbitrary variable loads is considered. The position of the crack tip is given as an arbitrary monotonically increasing differentiable function of time such that the velocity of crack propagation at any time is less than the Rayleigh wave velocity. An expression is obtained for the stresses on the crack plane ahead of the tip, particularly the stress intensity factors at its tip.

A fracture criterion permitting determination of the law of crack tip propagation under given external conditions is used to analyze crack propagation in fracture mechanics. In particular, the Griffith energy criterion which can be written as [1]

$$2\gamma(v) = -\frac{\pi}{2\mu b^2 v R(1/v)}\left(\sqrt{v^{-2}-a^{-2}}\,k_1^2 + \sqrt{v^{-2}-b^{-2}}\,k_2^2\right) + \frac{\pi}{2\mu v\sqrt{v^{-2}-b^{-2}}}k_3^2 \quad (0.1)$$

$$R(s) = (2s^2 - b^{-2})^2 + 4s^2\sqrt{a^{-2}-s^{-2}}\sqrt{b^{-2}-s^{-2}}$$

can be used for an ideally brittle, linearly elastic medium.

Here μ is the shear modulus, a and b are the longitudinal and transverse wave velocities, v is the velocity of crack propagation, $\gamma(v)$ is the effective surface energy which is considered a characteristic function of the crack propagation rate for a given material, and k_1, k_2, k_3 are the stress intensity factors for the three main modes of fracture: tensile, in-plane shear, and anti-plane shear (longitudinal shear), respectively. The function $R(s)$ vanishes at the points $s = \pm c^{-1}$, where c is the Rayleigh wave velocity.

In order to apply the criterion (0.1) to a specific problem, the stress intensity factors k_i must be known as functionals of the crack tip motion for which the solution of the corresponding dynamic problem of elasticity theory must be obtained for an arbitrary crack tip motion. This has been done in [2] for the particular case of an anti-plane shear crack. This case is simplest since only transverse waves polarized parallel to the crack edge originate. Recently Freund [3] found an expression for the intensity factor for a semi-infinite tensile crack being propagated at piecewise-constant velocity under the effect of static loads by using a clever semi-inverse method. Considering propagation at an arbitrary variable velocity as the limit case of a piecewise-constant velocity, he arrived at the deduction that the expression he obtained is valid even in the general case. The Freund result possesses two disadvantages. Firstly, this result has no foundation .